Nonlinear Flame-Flow Transfer Function Calculations: Flow Disturbance Celerity Effects

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Abstract
This paper describes an analysis of the nonlinear dynamics of premixed flames subjected to harmonic velocity disturbances. It generalizes our prior study, which assumed a spatially uniform velocity field, to allow for axial disturbance velocity variations, such as convected disturbances of arbitrary phase velocity. We show that the linear and nonlinear characteristics of the flame dynamics are controlled by the superposition of two sources of flame disturbances: those originating at the flame anchoring point due to boundary conditions and from flow non-uniformities. These disturbances do not generally propagate along the flame at the same speed. Consequently, they may either constructively or destructively superpose with each other depending upon frequency, flame length and radius, and the ratio of their propagation velocities. Though the two contributions to flame disturbance amplitude individually decrease with frequency at a fixed velocity disturbance amplitude, their sum has oscillatory behavior. In cases where they constructively interfere, the linear transfer function, \((A'/A_o)/(u'/u_o)\), exceeds unity in some cases.

In our prior study, which only included the effects of flame disturbances originating at the anchoring point, we showed that the flame area fluctuations do not increase proportionally to the disturbance velocity amplitude, resulting in a reduction in the nonlinear transfer function gain relative to its linear value. We show here that, because of the superposition of the two flame disturbance contributions, the flame transfer function relative to its linear value exhibits characteristics that depend upon whether the two solutions lie in a region of constructive or destructive interference. In regions of constructive interference, similar conclusions as in our prior study are obtained; i.e., the nonlinear flame transfer function gain is always less than its linear value. In cases where the two solutions lie in a region of destructive interference and are affected unequally by nonlinearity, the nonlinear transfer function exceeds its linear value. These predictions are shown to be consistent with recent measurements of Durox et al.32.

1. Introduction
This paper describes an analysis of the nonlinear dynamics of premixed flames responding to harmonic velocity disturbances. This work is motivated by the fact that current and next generation low emissions combustion systems for land-based gas turbines, possibly aircraft engines, and industrial boilers rely on a premixed or partially premixed combustion process. These systems are exceptionally prone to combustion instabilities1,2,3 which generally occur when the unsteady combustion process couples with the natural acoustic modes of the combustion chamber, resulting in self-excited oscillations. These oscillations are destructive to engine hardware and adversely affect engine performance and emissions.

A combustor’s dynamics are controlled by a complex interplay of linear and nonlinear processes. To illustrate, consider an acoustic disturbance with amplitude, \(\varepsilon\). Referring to Figure 1, note that this disturbance amplitude stays the same, decreases, or increases depending upon the relative magnitudes of the driving, \(H(\varepsilon)\), and damping, \(D(\varepsilon)\), processes; i.e., whether \(H(\varepsilon)=D(\varepsilon)\), \(H(\varepsilon)<D(\varepsilon)\), or \(H(\varepsilon)>D(\varepsilon)\), respectively. Linear combustor processes generally control the balance between driving and damping

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processes at low amplitudes of oscillation and, thus, determine the growth rate of inherent disturbances in the combustor. Nonlinear combustor processes control the finite amplitude dynamics of the oscillations. Predicting the limit cycle amplitude of self-excited oscillations requires an understanding of the nonlinear characteristics of \( H(\varepsilon) \) and \( D(\varepsilon) \). To illustrate, Figure 1 depicts a situation where \( H(\varepsilon) \) saturates and the two curves cross at the limit cycle amplitude, \( \varepsilon_{LC} \).

![Figure 1](image_url)

**Figure 1** Qualitative description of the dependence of acoustic driving, \( H(\varepsilon) \) and damping, \( D(\varepsilon) \), processes upon amplitude, \( \varepsilon \).

The focus of this paper is to characterize the heat release nonlinearities, i.e., to model the characteristics of \( H(\varepsilon) \). Linear heat release dynamics, while far from a solved problem, is better understood and has received extensive attention from academics and, more recently, by manufacturers for the development of dynamics predictions codes. In contrast, nonlinear combustor dynamics is significantly less developed and have not, to date, been included in the majority of the prediction codes being developed by industry (with the exception of Refs. [5] and [6]).

Most of the theoretical work in this area through the 1980’s focused upon nonlinear gas dynamical processes, i.e., \( D(\varepsilon) \). This work was primarily motivated by instability problems in rockets where fluctuating pressure amplitudes achieve significant percentages of the mean, e.g., \( p'/p_0 \sim 20-50\% \). Our focus on heat release dynamics is motivated by observations that the nonlinear gas dynamical processes are less significant in many premixed combustors. For example, Dowling suggests that gas dynamic processes essentially remain in the linear regime, even under limit cycle operation, and that it is the relationship between flow and heat release oscillations that provides the dominant nonlinear dynamics in premixed combustors, i.e., \( H(\varepsilon) \). The primary point of these observations have been confirmed by several experimental studies, which show that substantial nonlinearities in the heat release response to acoustic disturbances occurs, even at amplitudes as low as \( p'/p_0 \sim 1\% \) and \( u'/u_0 \sim 20\% \).

A variety of mechanisms exist for causing nonlinearities in heat release dynamics; e.g., local or global flame extinction, flame holding and/or nonlinear boundary conditions (e.g., the point where the flame anchors depending upon amplitude), equivalence ratio oscillations, and flame kinematics. It is this latter source of nonlinearity that is the focus of this study.

Premixed flame propagation normal to itself in an oscillatory flow field leads to substantial nonlinearities that are purely kinematic in origin. The groundwork for modeling the linear kinematical dynamics of premixed flames was laid by Marble and Candel and has proceeded rapidly through work at CNRS, MIT, and École Centrale Paris. Several key results on these linear dynamics are summarized next. The basic modeling approach follows from solving the front tracking equation for the flame position, commonly known as the G-Equation:

\[
\frac{\partial G}{\partial t} + \mathbf{u} \cdot \nabla G - S_L |\nabla G| = 0
\]  

(1)

where \( G(x,t) = 0 \) is an implicit expression defining the instantaneous flame position, \( \mathbf{u} \) is the velocity field and \( S_L \) is the laminar flame speed. Although not necessarily true in general, we assume from this point forward that the flame speed is fixed. An example where such an approximation is not appropriate occurs in the case of equivalence ratio oscillations, or at locations of flame cusps. As such, the flame’s heat release is directly related to its surface area, whose dynamics we specifically focus upon in this paper.

From a mathematical point of view, the linear solution to the equation for flame surface area can be decomposed into two canonical components: the homogeneous solution, due to the boundary conditions, and the particular solution, due to spatial non-uniformities in flow forcing or flame speed. In the reference frame of the flame, the linearized version of Eq. (1) can be written as (assuming constant \( S_L \))

\[
\frac{\partial \xi}{\partial t} + u \cdot \frac{\partial \xi}{\partial X} = u' (X,t)
\]  

(2)

where \( X \) denotes the coordinate along the flame front and \( \xi(X,t) \) is the perturbed flame position. The dynamics of \( \frac{\partial \xi}{\partial X} \), which is directly related to that of

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the flame area itself are described by the following solution:

\[
\frac{\partial \xi}{\partial X} = \int_0^X \frac{\partial}{\partial X} \left[ u'(x', t - \frac{X - x'}{u_o}) \right] \, dx' + \frac{1}{u_o} u'(X = 0, t - \frac{X}{u_o})
\]

The homogeneous and particular solutions have a clear physical significance which can be understood as follows. A spatially uniform disturbance velocity disturbance only excites the homogeneous solution (second term in Eq (3)). This can be understood by first assuming that the flame edge moves exactly in step with the particle velocity. In this case, the entire flame will simply move up and down in a bulk motion without a change in shape or area. However, if a flame anchoring boundary condition is imposed, such that the flame remains fixed, the flow disturbance excites a flame front disturbance that originates at the boundary and propagates along the flame front at a speed that is proportional to the mean flow velocity. These “homogeneous solution” flame dynamics were extensively analyzed by Fleifel et al.24

If the disturbance flow field is spatially non-uniform, the particular solution is excited (first term in Eq (3)). This results in waves originating at the spatial location(s) of flow non-uniformity that also propagate along the flame at roughly the mean flow velocity. Because the G-equation is first order in time, the flame acts as a low pass filter to flow disturbances, so that the amplitude of the two canonical solutions individually decay with increases in frequency as 1/f, but generally does not become identically zero. As such, the transfer function relating the response of the flame area to a spatially uniform velocity disturbance (where only the homogeneous solution is excited), \( \frac{A'/A_o}{u'/u_o} \), has a value of unity at zero frequency, decays monotonically with frequency, but generally is not identically zero. In contrast, when the flame is perturbed by a spatially non-uniform disturbance (so that both the homogeneous and particular solution are excited), the flame area consists of a superposition of the two solutions. As such, though each solution decreases with frequency, their sum has oscillatory behavior in cases where they constructively interfere, and even cause the transfer function, \( \frac{A'/A_o}{u'/u_o} \), to exceed unity. This result was first predicted by Schuller et al.\(^\text{31} \). In addition, the two solutions can destructively interfere, and in certain cases, exactly cancel each other so that the resulting transfer function \( \frac{A'/A_o}{u'/u_o} \) identically equals zero.

Consider next several basic features of the nonlinear flame dynamics. The key mechanism of nonlinearity is illustrated in Figure 2. In this illustration, a flame is perturbed by a transient disturbance so that it has a corrugated shape, but then allowed to relax back to its steady state, planar position. Flame propagation normal to itself smooths out the wrinkle, so that its area eventually returns to being constant in time. As such, kinematic processes work to destroy flame area, as shown by the dashed lines in the bottom sketch. The rate of these area destruction processes depends nonlinearly upon the amplitude of the flame front disturbance. Large amplitude corrugations are smoothed out at a relatively faster rate than small amplitude perturbations. In the same way, short length scale corrugations are smoothed out faster than long length scales. As discussed further below, this is the reason that nonlinearity is enhanced at higher disturbance frequencies, which generate shorter length scale flame corrugations.

Figure 2. Sketch of a flame that is initially wrinkled (top), showing the destruction of flame area by kinematic restoration processes (bottom)

The nonlinear dynamics of a flame responding to a uniform velocity field (so that only the homogeneous solution was excited) were discussed extensively in our prior study. It was noted that nonlinear effects always cause the nonlinear transfer function relating flame area and velocity perturbations, \( \frac{A'/A_o}{u'/u_o} \), to monotonically decrease with disturbance amplitude. In other words, the linear transfer function is always larger than the nonlinear transfer function. Since the scale of flame wrinkling is inversely proportional to frequency (scaling roughly as \( u_o/f \)), this reduction in

\(^1\) An exception occurs in two-dimensional flames at frequencies where the flame tip motion is zero. In this case, the flame’s linear area response is also zero.
finite amplitude transfer function relative to its linear value grows with frequency. As such, the flame area response to a velocity disturbance exhibits saturation characteristics, quite similar to the \( H(\varepsilon) \) curve plotted in Figure 1.

The objective of the present study is to generalize our prior study\(^1\) to cases where the velocity is not spatially uniform. Physical considerations suggest that the basic conclusions described above for the nonlinear cases should equally apply to the homogeneous or particular solution alone. However, it can be anticipated that, as in the linear case, the superposition of these solutions may result in different results, depending upon whether the two solutions lie in a region of constructive or destructive interference. In particular, it can be anticipated that if the two solutions lie in a region of destructive interference and are affected unequally by nonlinearity, their superposition may cause the nonlinear transfer function to actually exceed its linear value! We will show that this actually occurs in some cases and, furthermore, has apparently been experimentally observed.

Before proceeding to the analysis and results discussion, we consider briefly the characteristics of the non-uniform disturbance field that is perturbing the flame. A number of linear flame dynamics analyses assumed that the disturbance field was purely due to acoustic waves (as opposed to vorticity waves). In this case, the acoustic wavelength in many practical applications is often much longer than the flame length. As such the disturbance field may be assumed to be nearly uniform, as in Fleifel et al.’s\(^{24}\) analysis. Although two-dimensional effects cause some non-uniformities in the acoustic field, Lee and Lieuwen’s\(^{19}\) computations showed that the resulting solutions are still quite close to those derived from Fleifel et al.’s much simpler approach.

Experimental studies have clearly shown that the axial disturbance field is often not uniform, particularly in phase, and exhibits variation over a convective wavelength, \( u_0/\theta \). In Schuller et al.\(^{20}\) and Durox et al.’s\(^ {32}\) study, visualizations of the velocity field indicate that this is due to a convected vortical disturbance, excited at the shear layer of the burner exit. A photograph from Durox et al.’s study is reproduced in Figure 3, which superposes an image of the instantaneous wrinkled flame front and the convected vorticity field. By incorporating this convective phase variation into the disturbance velocity field, they show that the modeled flame area response agrees quite well with their data.

In general, however, it must be recognized that the disturbance field may have both acoustic and vortical components, whose relative magnitudes depends strongly upon the vortex shedding dynamics at the burner shear layer. These vortex shedding characteristics, in turn, are a function of the specific characteristics of the burner exit shear layer, such as co-flow velocity, and specifically upon the receptivity of this shear layer to external disturbances. For example, it is conceivable to think of an experiment with an appropriate co-flow where acoustic oscillations excite negligible convected vortical disturbances, so that the upstream disturbance field remains purely acoustic. Furthermore, even in cases where convected vortical disturbances are excited, their phase speed (celerity) is not necessarily equal to the flow velocity (as assumed in Schuller et al.’s\(^ {31}\) model), but varies with frequency and shear layer characteristics. In addition, the growth rate of these disturbances similarly varies with frequency and the shear layer characteristics.

![Figure 3 Vorticity field superposed with the flame front. Image reproduced with permission from D. Durox\(^ {32}\).](image)

To illustrate, Figure 4 plots Michalke’s\(^ {21}\) theoretical curves for the dependence of the phase speed, \( C_{ph} \), of shear layer instability waves in a jet flow upon Strouhal number, defined as \( S_\theta = f/\theta/\overline{u_o} \), for several values of the momentum thickness, \( \theta \), to jet radius, \( R \), ratio, \( R/\theta \). The figure shows that, for all \( R/\theta \) values, the ratio of \( C_{ph}/\overline{u_o} \) equals unity and 0.5 for low and high Strouhal numbers. For thin boundary layers, e.g., \( R/\theta =100 \), the phase velocity actually exceeds the maximum axial flow velocity. This ultra-fast phase velocity prediction has been experimentally verified by Bechert and Pfizenmaier\(^{22}\) and may explain a similar measurement in a Bunsen flame by Ferguson et al.\(^ {27}\).
This variation in vorticity wave growth rate with frequency may also explain the measurements of Ferguson et al.\textsuperscript{27} that the disturbance field at the flame shifts from being convected to acoustic in character at low and high frequencies, respectively.

For these reasons, we derive in this study a linear expression that generalizes the result of Schuller et al.\textsuperscript{31}, by determining the response of a flame to a disturbance with an arbitrary phase velocity.

Even in the absence of convected vorticity waves, the impact of the fluctuating flame position upon the acoustic field causes the acoustic disturbance field to have a convected character. This is due to the fact that the flame response to the acoustic field and the acoustic field disturbing the flame are coupled. For large amplitude disturbances, the flame develops large amplitude corrugations, such as can be seen in images from Durox in Figure 3, that convect with a phase speed proportional to the axial flow velocity. These convecting flame wrinkles impact the character of the interior acoustic field. In this case, it can be anticipated that the acoustic field structure reverts from being nearly uniform (assuming a compact flame) to having a convected character at low and high amplitude disturbances, respectively. Because we prescribe, rather than solve for, the disturbance field in this study, this effect, while potentially significant, is not captured in this analysis.

2. Modelling Approach

Figure 5 illustrates the two basic geometries considered. On the left is a conical flame stabilized on a tube, such as a Bunsen flame. On the right is an axisymmetric wedge flame, stabilized on a bluff body. The flame’s have axial and radial dimensions given by the flame length, $L_f$, and radius, $R$. The instantaneous flame sheet location at the radial location, $r$, is given by $\zeta(r,t)$, assumed to be a single-valued function of $r$. This assumption necessarily limits the range of amplitudes which can be treated with this formulation.

\textbf{2.1 Formulation}

The analytical approach used here closely follows DuCruix et al.\textsuperscript{23}, Baillot et al.\textsuperscript{16} and Fleifel et al.\textsuperscript{24}. Their work has shown good agreement between measured and predicted flame shapes and linear transfer functions. The flame’s dynamics are modeled with the front tracking equation\textsuperscript{25}:

\begin{equation}
\frac{\partial \zeta}{\partial t} = u - v \frac{\partial \zeta}{\partial r} - S_L \sqrt{\left( \frac{\partial \zeta}{\partial r} \right)^2 + 1}
\end{equation}

where $u$ and $v$ denote the axial and radial velocity components, and $S_L$ the flame speed.

In order to make analytical progress, the velocity is assumed to be (1) purely axial and (2) the flame speed constant. Regarding assumption (1), although the radial velocity is clearly not zero in reality, measurements and computations indicate that the velocity is dominated by the axial component, except possibly near the base of the flame\textsuperscript{26,27}. Because of assumption (2), the flame assumes sharp cusps in a few locations; if variable flame speed effects were included these cusps would smooth out over a length scale on the order of the flame thickness. We should emphasize that conditions can be readily identified where these two assumptions break down. However, our objective here is to address what is, in our view, a
foundational problem that captures the effects of kinematic nonlinearities on the flame’s area perturbations. More realistic but complicating influences should necessarily be considered, but not until the more basic problem has been treated and is thoroughly understood.

The variables t, r, a, u and ζ are non-dimensionalized by u_o/L_f, R, R, u_o and L_f (note that the value of L_f and R refer to their nominal values without imposed oscillations), where u_o is the mean axial velocity. They are related to the flame speed and average flow velocity by:

\[ \frac{u_o}{S_t} = \left( \frac{L_f}{R} \right)^{\frac{1}{2}} + 1 \]  

(5)

The ratio of the flame length to radius plays an important role in the flame’s dynamics and is denoted by β.

\[ \beta = \frac{L_f}{R} \]  

(6)

The velocity field is given as:

\[ u(\zeta, t) = u_0 + u_0 \cos(k_0 \zeta - \omega_0 t) \]  

(7)

Here the convective wave number k is defined as:

\[ k = \frac{\omega_0}{v_c} = \left( \frac{\omega_0}{v_0} \right) \left( \frac{u_o}{v_0} \right) = K \left( \frac{\omega_0}{u_0} \right) \]  

(8)

where u_c is defined as the phase velocity of the disturbance and \( \omega_0 \) denotes the angular frequency of the velocity disturbance. K is a parameter which denotes the ratio of the mean flow velocity to the phase velocity of the disturbances.

Given these assumptions, the flame dynamics are given by (from this point on we use the same symbol for the dimensionless variable):

\[ \frac{\partial \zeta}{\partial t} + \sqrt{1 + \beta^2 \left( \frac{\partial \zeta}{\partial t} \right)^2} = u(\zeta, t) \]  

(9)

The non-dimensionalized velocity field is given by:

\[ u(\zeta, t) = 1 + \varepsilon \cos[\text{St}(K_0 \zeta - t)] \]  

(10)

where

\[ \text{St} = \frac{\omega_0 L_f}{u_o} \]

Velocity perturbation: \( \varepsilon = \frac{u}{u_o} \)

The initial flame shape is given by

\[ \zeta(r, 0) = 1 - r \]  

(11)

Following prior studies we assume that the flame remains anchored at the base; i.e.,

\[ \zeta(r = 1, t) = 0 \]  

(12)

This boundary condition cannot be used for disturbance velocity magnitudes where the instantaneous flow velocity is lower than the flame speed. In this case, the flame will flash back and Eq. (12) must be replaced by a different condition; e.g., see Dowling [13]. In this study, calculations are performed only for velocity magnitudes lower than this critical value, which will be referred to as \( \varepsilon_f \), where:

\[ \varepsilon_f = 1 - \frac{1}{\sqrt{1 + \beta^2}} \]  

(13)

2.2 Computational Approach

This section describes the numerical approach used to solve Eq. (9) (an analytical perturbation analysis carried out to third order in amplitude will be described in a companion paper). A robust numerical scheme is necessary which can accurately capture the formation of sharp gradients and cusps in the distorted flame front. Spatial derivatives are discretized using a Weighted Essentially Non-Oscillatory (WENO) scheme designed specifically for Hamilton-Jacobi equations. This scheme is uniformly fifth order accurate in space in the smooth regions and third order accurate in discontinuous regions. Derivatives at the boundary nodes are calculated using fifth order accurate upwind-differencing schemes so that only the nodes inside the computational domain are utilized. A Total Variation Diminishing (TVD) Runge-Kutta scheme, up to third order accurate, is used for time integration. The flame front perturbation is tracked and the corresponding change in the flame surface area is calculated as a function of time for a given upstream flow velocity perturbation. The transfer function relating the flame area to the convective velocity perturbation is then evaluated.

3. Results and Discussion

The flame front tracking equation Eq. (9) describes the kinematics of either the conical or wedge shaped flames. Thus, given the assumptions of this analysis, the dynamics of the flame front itself is the same for
both geometries considered here. The dynamics of the area response differ, however. In the next section, we consider the linear response of these flame’s to convected disturbances with arbitrary phase speeds. The following section then generalizes these results to the fully nonlinear case.

3.1 Linear Response

In this section, we derive an expression for the flame area-velocity transfer function that generalizes the result of Schuller 31, by determining the response of a flame to a disturbance with an arbitrary phase velocity. The analytical solution in the linear regime can be expressed as 16:

\[
\zeta(r,t) = \zeta_0(r) + \zeta_1(r, t) + O(e^2) \quad (14)
\]

The flame shape in the absence of perturbations is given by:

\[
\zeta_0 = 1 - r; \quad (15)
\]

The evolution equation for \(\zeta_1\) is:

\[
\frac{\partial \zeta_1}{\partial t} - \alpha \frac{\partial \zeta_1}{\partial r} - \cos \left[ St \left\{ K(1-r) - t \right\} \right] = 0 \quad (16)
\]

where \(\alpha = \frac{\beta^2}{\beta^2 + 1}\)

The solution of Eq.(16), given the boundary condition in Eq.(12), is:

\[
\begin{aligned}
\zeta_1 & = \zeta_{1,BC} + \zeta_{1,Flow} \\
\sin \left[ \frac{St \left\{ (r-1)+\alpha t \right\}}{\alpha} \right] & = \sin \left[ St \left\{ K(r-1)+t \right\} \right] \\
\sin \left[ \frac{St(r-1)-1}{2\alpha} \right] & = 2 \frac{\sin \left[ \frac{St(r-1)}{2\alpha} \right]}{\sin \left[ \frac{St(r-1)}{2\alpha} \right]} \\
\cos \left[ \frac{(\eta+1)(r-1) + t}{\alpha} \right] & = 2 \frac{(\eta+1)(r-1) + t}{\alpha} \\
\end{aligned} \quad (17)
\]

where

\[
\eta = K\alpha \quad (18)
\]

This equation explicitly decomposes the solution into the contributions from the boundary conditions and flow forcing non-uniformities.

The flame front position is controlled by the superposition of the flame disturbances created at each point due to flow non-uniformity and disturbances originating from the flame base that are convected along the flame front. In the limit where \(\eta \to 1\), the solution is given by:

\[
\lim_{\eta \to 1} \zeta_1 = \left( \frac{1-r}{\alpha} \right) \cos \left[ St \left\{ \frac{(r-1)}{\alpha} + t \right\} \right] \quad (19)
\]

Consider next the instantaneous flame surface area. The non-dimensionalized surface area for a conical flame is given by:

\[
A_c(t) = \frac{1}{A_{c,0}} \frac{\int_0^1 \sqrt{1 + \beta^2 \left( \frac{\partial \zeta}{\partial r} \right)^2} \, dr}{\sqrt{1 + \beta^2}} \quad (20)
\]

Substituting Eqs. (14),(15) and (17) in Eq. (20) and defining

\[
G_c = \frac{A_c'}{A_{c,0}} \quad (u' / u_0) \quad (21)
\]

where

\[
G_c(St_1, \eta) = G_{c,BC} + G_{c,Flow} = 2 \left[ \frac{1}{\eta(\eta - 1)} \left( \frac{\exp(\text{i} St_1) - 1}{\exp(\text{i} St_1) + \text{i} \exp(\text{i} St_1)} \right) \right] + 2 \left[ \frac{1}{\eta(\eta - 1)} \left( \frac{\exp(\text{i} St_1)}{\eta(\eta - 1)} \right) \right] \quad (22)
\]

The non-dimensionalized surface area for the wedge flame attached at a point is given by (refer Figure 5):

\[
A_w(t) = \frac{1}{A_{w,0}} \frac{\int_0^1 \sqrt{1 + \beta^2 \left( \frac{\partial \zeta}{\partial r} \right)^2} \, dr}{\sqrt{1 + \beta^2}} \quad (23)
\]

Substituting Eqs. (14),(15) and (17) in Eq. (23) and defining

\[
G_w = \frac{A_w'}{A_{w,0}} \quad (u' / u_0) \quad (24)
\]

where

\[
G_w(St_2, \eta) = G_{w,BC} + G_{w,Flow} = 2 \left[ \frac{1}{\eta(\eta - 1)} \left( \frac{\exp(\text{i} St_2) - 1}{\exp(\text{i} St_2) + \text{i} \exp(\text{i} St_2)} \right) \right] + 2 \left[ \frac{1}{\eta(\eta - 1)} \left( \frac{\exp(\text{i} St_2)}{\eta(\eta - 1)} \right) \right] \quad (25)
\]
Thus, the linear flame transfer functions for both the conical and wedge flames, Eq. (21) & (24), only depend on two parameters: $\beta$ and $\eta$. The term $\eta$ couples the effect of flame angle and phase speed of the disturbances. These expressions reduce to those derived by Schuller et al\textsuperscript{31} when the phase speed of the disturbances is equal to that of the mean flow (i.e. $K=1$) (note that they refer to $St_2$ as $\omega_*$ and $\alpha$ as $\cos^2 \alpha$). As noted earlier, the value of $K$ is sensitive to the characteristics of the burner exit shear layer and the forcing frequency. For example, Baillot et al\textsuperscript{16} measured $K$ values of 0.88 and 0.98 at 35 and 70 Hz, respectively, on a conical Bunsen flame. Similarly, Durox et al\textsuperscript{32} measured $K=2$ values at 150 Hz in an axisymmetric wedge flame.

It is useful to define a Strouhal number ($St_c$) based upon the convective velocity ($u_c$) of the flow disturbances. $St_c$ naturally arises in the two transfer functions (Eq.(21) & Eq (24)) and equals $\eta St_2$:

$$\eta St_2 = K \times St = \left(\frac{u_c}{u_o}\right) \left(\frac{\omega_o}{\omega_*}\right) = \frac{\omega_o L_F}{u_c} = St_c \quad (25)$$

These two Strouhal numbers are related to the amount of time taken for a flow ($St_c$) and flame front ($St_2$) disturbance (which is ultimately created by a flow disturbance) to propagate the flame length, normalized by the acoustic period.

Before looking at the total flame transfer functions, it is useful to understand the characteristics of its two contributing flow forcing and the boundary condition terms. Their ratio is given by:

$$\frac{G_{c,Flow}}{G_{c,BC}} = \frac{1 - \exp(i \eta St_2) + i \eta St_2}{\eta \left[\exp(i St_2) - 1 - i St_2\right]} \quad (26)$$

$$\frac{G_{w,Flow}}{G_{w,BC}} = \frac{(1 - i \eta St_2) \exp(i \eta St_2) - 1}{\eta \left[1 - (1 - i St_2) \exp(i St_2)\right]} \quad (27)$$

Interestingly, the magnitude of this ratio is identical for both wedge and conical flames, see Figure 6. The phase of this ratio is different for conical and wedge flames and plotted in Figure 7 and Figure 8, respectively.

It is instructive to analyze the characteristics of this ratio for limiting values of the parameters $\eta$ and $St_2$. First, note that in the $\eta \rightarrow 0$ limit (i.e., a spatially uniform disturbance), the flame dynamics for both the wedge and conical flames is controlled exclusively by the boundary condition term, irrespective of Strouhal numbers.

$$\lim_{\eta \rightarrow 0} \left(\frac{G_{c,Flow}}{G_{c,BC}}\right) = \lim_{\eta \rightarrow 0} \left(\frac{G_{w,Flow}}{G_{w,BC}}\right) = 0 \quad (28)$$

This result can be anticipated from the discussion in the Introduction section and reflects the fact that only the homogeneous solution is excited when the flow disturbance is uniform.

$$\lim_{St_2 \rightarrow 0} \left(\frac{G_{c,Flow}}{G_{c,BC}}\right) = \lim_{St_2 \rightarrow 0} \left(\frac{G_{w,Flow}}{G_{w,BC}}\right) = -\eta \quad (29)$$

The boundary condition and flow forcing terms dominate when $\eta<1$ and $\eta>1$, respectively. For long flames ($\beta \gg 1$), this physically corresponds to situations where the disturbance phase velocity is greater than and less than the mean flow velocity, respectively. The two terms tend toward equal magnitudes when $\eta=1$. These points can be clearly observed in Figure 6. Note also that the flow disturbance and boundary condition terms are 180° out of phase for low $St_2$ values see Figure 7 and Figure 8.

$\eta=1^1$. Some care is required in interpreting this $\eta=1$ result, as the two terms tend to have equal magnitudes and are 180 degrees out of phase. The overall response is not zero, however, as the common denominator ($\eta-1$) in Eqs. (21) and (24), which has been cancelled out when taking their ratio, causes their sum to have a non-zero value.

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Figure 7 Strouhal number dependence of the phase of the ratio of the transfer functions due to the flow forcing and boundary condition terms for conical flames for different values of $\eta$.

In the $St_2 >> 1$ limit, the contribution from both the boundary conditions and flow forcing term are equal, as shown in Figure 6 and below:

$$\lim_{St_2 \to \infty} \left( \frac{G_{c,\text{Flow}}}{G_{c,\text{BC}}} \right) = -1$$

$$\lim_{St_2 \to \infty} \left( \frac{G_{w,\text{Flow}}}{G_{w,\text{BC}}} \right) = -\exp[i(\eta-1)St_2]$$

Equation (30) also shows that, in this limit, the relative magnitude contribution of these two terms is independent of $\eta$ (assuming that the $\eta St_2$ product does not simultaneously go to zero). Moreover, the two terms are always out of phase for conical flames irrespective of the Strouhal number and $\eta$, as shown in Figure 7. In contrast, for wedge flames the phase difference between the two contributions monotonically increases with $St_2$, as shown in Figure 8 (the shaded bands in the figure indicate regions of constructive interference). In the intermediate Strouhal number range, say $0.1 < St_2 < 20$, either the flow forcing or the boundary condition may dominate depending upon $\eta$ and $St_2$.

Figure 8 Strouhal number dependence of the phase of the ratio of the transfer functions due to the flow forcing and boundary condition terms for wedge flames for different values of $\eta$. Shaded regions indicate points where boundary condition and flow forcing terms are in phase.

The dependence of the magnitude and phase of the conical flame transfer function $G_c(St_2, \eta)$ upon $St_2$ at several $\eta$ values is plotted in Figure 9 and Figure 10, respectively. Consider the magnitude results first. Note that the transfer function gain is identical in the cases where $\eta = 0$ or 1. Physically, this corresponds to cases where the disturbance velocity is uniform ($\eta = 0$) or its phase speed matches the flame front disturbance velocity ($\eta = 1$). This equality of gain transfer functions was previously noted by Schuller et al.\textsuperscript{31} The gain transfer function differs for all other disturbance phase velocity cases. Note also that the gain value is always less than one and generally decreases monotonically with $St_2$, although there is some ripple at higher $St_2$ values due to constructive and destructive interference between $G_{c,\text{Flow}}$ and $G_{c,\text{BC}}$. The transfer function phase starts at zero degrees at low $St_2$ and initially increases monotonically with $St_2$. In the $\eta = 0$ case, the phase tends to a limiting value of 90º for large $St_2$. In all other cases, the phase monotonically increases and for high values of $\eta$ and $St_2$ the phase curves collapse into a single line.

Figure 9 Axisymmetric conical linear transfer function $G_c(St_2, \eta)$ amplitude dependence upon the reduced Strouhal number ($St_2$) for different values of $\eta$. 

The dependence of the magnitude and phase of the conical flame transfer function $G_c(St_2, \eta)$ upon $St_2$ at several $\eta$ values is plotted in Figure 9 and Figure 10, respectively. Consider the magnitude results first. Note that the transfer function gain is identical in the cases where $\eta = 0$ or 1. Physically, this corresponds to cases where the disturbance velocity is uniform ($\eta = 0$) or its phase speed matches the flame front disturbance velocity ($\eta = 1$). This equality of gain transfer functions was previously noted by Schuller et al.\textsuperscript{31} The gain transfer function differs for all other disturbance phase velocity cases. Note also that the gain value is always less than one and generally decreases monotonically with $St_2$, although there is some ripple at higher $St_2$ values due to constructive and destructive interference between $G_{c,\text{Flow}}$ and $G_{c,\text{BC}}$. The transfer function phase starts at zero degrees at low $St_2$ and initially increases monotonically with $St_2$. In the $\eta = 0$ case, the phase tends to a limiting value of 90º for large $St_2$. In all other cases, the phase monotonically increases and for high values of $\eta$ and $St_2$ the phase curves collapse into a single line.

Figure 9 Axisymmetric conical linear transfer function $G_c(St_2, \eta)$ amplitude dependence upon the reduced Strouhal number ($St_2$) for different values of $\eta$. 

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Figure 10 Axisymmetric conical linear transfer function $G_c(S_{t2}, \eta)$ phase dependence upon the reduced Strouhal number ($S_{t2}$) for different values of $\eta$.

Figure 11 Axisymmetric wedge linear transfer function $G_w(S_{t2}, \eta)$ amplitude dependence upon the reduced Strouhal number ($S_{t2}$) for different values of $\eta$.

For wedge flames, the gain and phase of the flame transfer function $G_w(S_{t2}, \eta)$ are shown in Figure 11 and Figure 12 respectively. Note that all gain values tend toward values of unity at low $S_{t2}$. However, only in the uniform velocity case, $\eta=0$, does the gain then decrease with increases in $S_{t2}$ as might be expected. In all other cases, the gain increases to values of greater than unity, due to constructive interference between $G_{w,\text{Flow}}$ and $G_{w,Bc}$. This amplification of the flame response over its quasi-steady value was previously pointed out by Schuler et al\textsuperscript{31}. The magnitude and $S_{t2}$ value of the peak value of this amplification region is controlled by $\eta$. As shown in Figure 11, the magnitude of the peak value of $G_w$ initially increases from unity as $\eta$ increases with zero, reaches a maximum at $\eta=1$, and then decreases back to unity with further $\eta$ increases. Note also from Figure 11 that the secondary maxima in the transfer functions appear at a lower $S_{t2}$ values with increases in $\eta$. This behavior can be understood by noting that increases in $\eta$ at a fixed $S_{t2}$ is equivalent to an increase in the convective Strouhal number, $S_{t2}=\eta S_{t2}$.

Figure 12 Axisymmetric wedge linear transfer function $G_w(S_{t2}, \eta)$ phase dependence upon the reduced Strouhal number ($S_{t2}$) for different values of $\eta$.

Another striking feature is the resonance-like behavior at $\eta=1$, where the wedge flame response does not decrease with $S_{t2}$ but tends toward a constant value of two. This case corresponds to exact coincidence of flame front and flow disturbance velocity. In reality, curvature effects on flame speed that are neglected in this analysis, which increase with $S_{t2}$, cause the transfer function to decrease at higher values of $S_{t2}$.

Turning to the phase in Figure 12, note that the phase increases with $S_{t2}$ with similar characteristics for all $\eta$ values.

The variation of the transfer functions with $\eta$ is plotted for conical (Figure 13) and wedge (Figure 14) respectively. For any intermediate value of $\eta$ the flame dynamics is controlled by competition between the flow disturbance non-uniformity effects and boundary conditions. These two effects can constructively interfere to give a higher gain or destructively interfere to give a reduction in the gain (i.e. the local maxima and minima in Figure 13 and Figure 14). This fact is further highlighted in Figure 15 which shows the Strouhal number dependence of the value of the parameter $\eta$ at which the maximum possible response occurs for conical and wedge flames. For conical flames in the range $0<S_{t2}<8$, the maximum gain occurs
at $\eta=0.5$ (see Figure 9, Figure 13 and Figure 15). For $St_2>8$, the maximum response occurs at two $\eta$ values ($\eta=0$ & $\eta=1$) as shown in Figure 15. For wedge flames, the maximum possible gain for $St_2>5$ occurs at $\eta=1$ (see Figure 14 and Figure 15). Note that for the $\eta=1$ case, the wedge flame acts as an amplifier for a wide range of $St_2$ values as shown in Figure 11. It is interesting to note that, for any Strouhal number, the peak response for wedge flames always occurs for $\eta \geq 1$; i.e. for cases wherein the phase speed of the disturbances is less than the mean flow speed. In contrast the maximum gain for conical flames, at any Strouhal number, always occurs for $\eta \leq 1$; i.e. for cases wherein the disturbances propagate faster than the mean flow speed.

We next turn to the response of the flame area in the general, nonlinear case. Before considering specific results, several general conclusions which can be obtained from analysis of the equations should be considered.

First, note that in the linear case the transfer function is described by only two parameters; i.e. $G_{Lin} = G(St_2, \eta)$. However for the general nonlinear case the gain $G$ is also dependent on $\varepsilon$ and $\beta$; i.e., $G = G(St_2, \eta, \varepsilon, \beta)$.

Next, note that nonlinearities in the flame area response arise from three sources. The first is the nonlinear flame dynamics, through the term $\frac{1}{\varepsilon^2} \left( \frac{\partial \zeta}{\partial \tau} \right)^2$ in Eq.(9). The second is the static nonlinearity introduced through the dependence of the flame area upon flame position gradient through a term with the same form, $\frac{1}{\varepsilon^2} \left( \frac{\partial \zeta}{\partial \tau} \right)^2$, see Eqs. (20) and (23). In both of these cases, the nonlinearity is purely geometric in origin and is introduced by the relationship between the instantaneous flame front normal and flame position gradient. The third nonlinearity is due to the flow forcing itself, and is due to the dependence of the disturbance velocity at the flame front upon the flame position, $u(\zeta, t)$.

The fact that the first two sources of nonlinearity are identical can be used to write the final expressions for the flame area, (Eqs. (20) & (23)), in a revealing form. By substituting Eq.(9) into Eqs. (20) & (23),
note that the term \( \sqrt{1+\beta^2 \left( \frac{\partial \zeta}{\partial \tau} \right)^2} \) which appears in both the area integrals can be written as:

\[
\sqrt{1+\beta^2 \left( \frac{\partial \zeta}{\partial \tau} \right)^2} = u(\zeta,t) - \frac{\partial \zeta}{\partial \tau}
\]  

(31)

Thus, the explicit form of the nonlinearity disappears. Nonlinearities in flame front dynamics are included in the \( \partial \zeta / \partial \tau \) term (note that the flow forcing nonlinearities also effect \( \partial \zeta / \partial \tau \), as shown in Eq.(9)), while those due to the flow forcing nonlinearity noted above are included in the \( u(\zeta,t) \) term. Based upon Eq. (31), the following observations can be made regarding the effects of various parameters upon nonlinearity in the flame’s response to flow perturbations.

1. Strouhal Number: At low Strouhal numbers, the unsteady term in Eq. (31) is negligible. Moreover, the \( \zeta \) dependence of the velocity field, \( u(\zeta,t) \), will be weak in the limit of low \( St_c \), at least for the velocity fields considered here. Thus, the flame area’s velocity response remains linear for low Strouhal numbers, as \( St \) is the dimensionless ratio of the flame response time to perturbation period. This point shows that the flame’s nonlinear area response is an intrinsically dynamic phenomenon; its quasi-steady response is linear.

2. Flow Uniformity: Nonlinearities in the \( u(\zeta,t) \) term are directly due to non-uniformity in flow disturbances. Thus, the contribution of this term to flame area nonlinearities is suppressed in the \( \eta \to 0 \) limit.

3. Boundary conditions: if the flame remains anchored at the attachment point, as it is in this study, then \( \partial \zeta / \partial \tau \) is identically zero at this point for all time. As such, the flame area perturbations in the vicinity of the attachment point (where \( \zeta \approx 0 \Rightarrow u(0,t) \)) exhibit a linear dependence upon velocity magnitude. Nonlinearities only arise at points of the flame that are spatially removed from the attachment point. As such, the axisymmetric conical flame exhibits a more linear velocity response than the axisymmetric wedge flame for comparable values of \( \varepsilon \), since most and very little, respectively, of the flame area is concentrated near the attachment point. This discussion also shows the potential coupling between the flame kinematic and flame holding sources of nonlinearity.

4. Flame Aspect Ratio: The flame dynamics tend to linearity when \( \beta \gg 1 \); i.e., when the flame is very long. This can be seen by noting that the left side of Eq (9) becomes linear in this case:

\[
\sqrt{1+\beta^2 \left( \frac{\partial \zeta}{\partial \tau} \right)^2} \approx \pm \beta \left( \frac{\partial \zeta}{\partial \tau} \right)
\]

As such, the flame dynamics is approximately described by the equation \( \partial \zeta / \partial \tau \pm \partial \zeta / \partial \tau = u(\zeta,t) \). In this case, the flame dynamics are linear, although the flow forcing term need not be. This discussion shows that \( \beta \) is an important nonlinearity parameter for this problem; i.e., the flame’s area response can be anticipated to exhibit a linear dependence upon the perturbation velocity for much larger \( \varepsilon \) values at large \( \beta \) values.

The rest of this section presents typical results comparing the linear and nonlinear flame transfer function. The nonlinear flame transfer function was determined by computing the flame area only at the forcing frequency (since higher harmonics are also excited) via the Fourier transform.

Figure 16-Figure 19 plot the \( St_2 \) dependence of the gain and phase of the nonlinear transfer function upon \( St_2 \) at several \( \eta \) values. The gain transfer functions are normalized by their linear values, \( G/G_{Lin} \). Results are shown for two convective wave speeds, \( \eta=0 \) & \( \eta=1 \), but the same \( \beta=2 \) value. Note that the \( \eta=0 \) case represents a spatially uniform velocity flow field whereas the \( \eta=1 \) case represents a flow disturbance whose phase speed is equal to the mean flow speed. Thus, these results allow for a convenient comparison of the effects of nonlinearities from boundary conditions alone, and the combined effect of boundary conditions and flow disturbance non-uniformity.

Consider the gain curves first, Figure 16. As predicted above, the response tends to its linear value in all cases at low \( St_2 \). In the \( \eta=0 \) case, more nonlinear
effects are apparent at high St2. For the wedge flame in Figure 18 the response is considerably nonlinear even at moderate values of Strouhal number. The enhanced nonlinear response of wedge over cone flames is explained by the Boundary Conditions argument above. Note that extensive results for this η=0 case are included in our prior publication17.

Turning to the η=1 case, note the substantial reduction in flame area relative to its linear value; i.e., there is a substantial degree of gain saturation. Although plotted in a different form, the resulting gain curves look quite similar to the qualitative plot of $H_n(A)$ in Figure 1. In agreement with the Strouhal Number argument above, the degree of nonlinearity increases with St2. As shown in Figure 16 & Figure 18, the gain for the conical and wedge flames decreases by about 80% at $\varepsilon=\varepsilon_f$ in the St2= 20 and η= 1 case. In contrast, the gain never drops below 5% of its linear value for conical flames and 45% of its linear value for wedge flames in the η=0 case. Moreover, unlike the η=0 case, the phase of the area response for both the conical and wedge flames exhibits a strong amplitude dependence, as shown in Figure 17 and Figure 19 respectively. These results indicate the extent to which flow non-uniformities can significantly modify the nonlinear flame response.
discussion in the context of Figure 2 shows that the magnitude of their gain reductions is different. Since the individual gain decreases by different amounts, the total gain does not go to zero at the St$_2$ value at which the linear gain is zero, but actually shifts to a higher St$_2$ value in the $\varepsilon=0.2\varepsilon_f$ case. At higher disturbance levels, the two terms never exactly cancel and the gain does not go to zero. Rather, there is a monotonic decrease in the gain of the transfer function with increase in velocity amplitude. Analogous behavior also occurs in conical flames, although less dramatically. A representative case is shown in Figure 21 for $\eta=0.5$ (i.e. disturbances are traveling at double the mean flow speed).

These predictions can be compared to related measurements of Durox et al.$^{32}$ on wedge flames, where it was observed that the phase speed of the disturbances was half the mean flow speed. They obtained measurements at four forcing amplitudes $\varepsilon \sim 0.05-0.2$ and $\beta \sim 5.6$. Interestingly, they found both increases and decreases in the transfer function gain with changes in disturbance amplitude, depending upon Strouhal number. Their results are reproduced in Figure 22. The transfer functions plotted here equal the ratio of the fluctuating CH$^*$ emission intensity to the velocity disturbance amplitude slightly above the burner exit. Note the strong similarities between their measurement and the predictions from Figure 20. In the $2<St_2<5$ regions where the nonlinear transfer function exceeds unity, the nonlinear gain monotonically decreases with disturbance amplitude. In the $5<St_2<8$ range, the nonlinear transfer function first increases with disturbance amplitude, then decreases. This trend is completely consistent with the predictions of this study. We should note that Durox et al. suggest that the relationship between the flame heat release and the chemiluminescence emissions may not be linear due to the unconfined nature of the flame and the entrainment of ambient air into the mixture. While this is certainly an important point, it is not clear why it would cause the transfer function crossover with disturbance amplitude at St$_2 \sim 5$.

The regions where the nonlinear transfer function is larger or smaller than its linear value can be visualized from Figure 23 and Figure 24, which plot the gain dependence upon $\eta$ at different St$_2$ values for conical and wedge flames. For the St$_2=1$ case, the linear gain does not go to zero for both the conical and wedge flames. Hence, the gain monotonically decreases with increasing disturbance amplitude. However in the $5<St_2<8$ range, the nonlinear transfer function first increases with disturbance amplitude, then decreases. This trend is completely consistent with the predictions of this study. We should note that Durox et al. suggest that the relationship between the flame heat release and the

![Figure 21](image1.png)

![Figure 22](image2.png)
the sensitivity of the flame response to the phase speed of the disturbances. Moreover, it demonstrates how the competition between the boundary condition and flow forcing terms can significantly impact the flame response behavior.

Figure 23 Dependence of the magnitude of the flame area-velocity transfer function on $\eta$ for the axisymmetric conical flame for different Strouhal numbers, $\beta=8$. Arrow points towards increasing velocity amplitude $\varepsilon$.

Figure 24 Dependence of the magnitude of the flame area-velocity transfer function on $\eta$ for the axisymmetric wedge flame for different Strouhal numbers, $\beta=8$. Arrow points towards increasing velocity amplitude $\varepsilon$.

**Concluding Remarks**

These results have implications of the type of bifurcations which may be observed in unstable combustors. In situations where the gain curves resemble that qualitatively shown in Figure 1, only supercritical bifurcations will occur and only a single stable limit cycle amplitude, $\varepsilon_{LC}$ is possible. In situations where the gain exceeds, then is less than, the linear gain, multiple stable solutions for the instability amplitude may exist, and sub-critical bifurcations are possible. This can be seen from Figure 25, which plots the dependence of $A'/A_o$ vs $\varepsilon$ for $St_2=6.25$, $\eta=2$, $\beta=8$. This curve represents $H(\varepsilon)$. A hypothetical damping curve, $D(\varepsilon)$ is also drawn in. Note the 3 intersection points, two of which are stable, $\varepsilon=0$ and $\varepsilon=\varepsilon_{LC}$, and one of which is unstable, $\varepsilon=\varepsilon_T$. In this case two equally valid solutions are possible, $\varepsilon=0$ or $\varepsilon=\varepsilon_{LC}$, which one the system is actually at depends upon initial conditions. Such a system will manifest characteristics such as hysteresis and triggering (i.e., the destabilization of a linearly stable system by a sufficiently large disturbance$^{33}$).

Figure 25 Dependence of acoustic driving, $H(\varepsilon)$ and damping, $D(\varepsilon)$, processes upon velocity amplitude $\varepsilon$, for wedge flames at $St_2=6.25$ with $\eta=2$ and $\beta=8$.

This study has highlighted the importance of the interactions between the contributions from flame disturbances due to boundaries and flow non-uniformities. To quantify the relative effects of nonlinearity upon these terms, as well as their nonlinear interactions, we have performed a perturbation analysis of (9) to determine the flame transfer function. This is a tedious calculation as a consistent approximation of the lowest order nonlinear correction to the transfer function requires carrying out the calculations to third order in perturbation amplitude. These calculations will be reported in Ref. [28].

References
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